Effective degrees of nonlinearity in a family of generalized models of two-dimensional turbulence

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We study the small-scale behavior of generalized two-dimensional turbulence governed by a family of model equations, in which the active scalar $\theta=(-\Delta)^{\alpha/2}\psi$ is advected by the incompressible flow $\boldsymbol{u}=(-\psi_y,\psi_x)$. Here ψ is the stream function, Δ is the Laplace operator, and α is a positive number. The dynamics of this family are characterized by the material conservation of θ, whose variance $\langle\theta^2\rangle$ is preferentially transferred to high wave numbers (direct transfer). As this transfer proceeds to ever-smaller scales, the gradient $\nabla\theta$ grows without bound. This growth is due to the stretching term $(\nabla\theta\cdot\nabla)\boldsymbol{u}$ whose "effective degree of nonlinearity" differs from one member of the family to another. This degree depends on the relation between the advecting flow $\boldsymbol{u}$ and the active scalar θ (i.e., on α) and is wide ranging, from approximately linear to highly superlinear. Linear dynamics are realized when $\nabla\boldsymbol{u}$ is a quantity of no smaller scales than θ, so that it is insensitive to the direct transfer of the variance of θ, which is nearly passively advected. This case corresponds to $\alpha\geq 2$, for which the growth of $\nabla\theta$ is approximately exponential in time and nonaccelerated. For $\alpha<2$, superlinear dynamics are realized as the direct transfer of $\langle\theta^2\rangle$ entails a growth in $\nabla\boldsymbol{u}$, thereby, enhancing the production of $\nabla\theta$. This superlinearity reaches the familiar quadratic nonlinearity of three-dimensional turbulence at $\alpha=1$ and surpasses that for $\alpha<1$. The usual vorticity equation ($\alpha=2$) is the border line, where $\nabla\boldsymbol{u}$ and θ are of the same scale, separating the linear and nonlinear regimes of the small-scale dynamics. We discuss these regimes in detail, with an emphasis on the locality of the direct transfer.

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I. INTRODUCTION

The production of progressively smaller scales, possibly to be limited by viscous effects only, in incompressible fluid flow at high Reynolds numbers is a fundamental problem in fluid dynamics. This long-standing problem is of genuine interest for obvious reasons. One is that the production of small scales plays a key role in the possible development of singularities from smooth initial conditions in the three-dimensional (3D) Euler or Navier-Stokes equations that govern the flow. Another reason is that in the presence of a large-scale forcing, a persistent production of small scales would be crucial to maintain a spectral energy flux (direct energy cascade). The realizability of such a steady and viscosity-independent flux is central to the Kolmogorov theory of turbulence as this would be required to rid the virtually inviscid energy inertial range of the injected energy, thereby, making it possible for a statistical equilibrium to be established. This dynamical scenario is either explicitly or implicitly assumed to apply to other fluid systems as well, not just the 3D Navier-Stokes equations. For example, in the Kraichnan-Batchelor [1–3] theory of two-dimensional (2D) turbulence, the dynamics of the mean-square vorticity (twice the enstrophy) are assumed to be synonymous in many aspects to those of the 3D energy. In particular, the enstrophy injected into the system at large scales is hypothesized to cascade to a dissipation range at small scales. As another example, the mean-square potential vorticity in the quasigeostrophic geophysical flow model is believed to behave in a similar manner [4]. Thus "cascading dynamics" have been considered universal among fluid systems.

The evolution of fluid flow is intrinsically nonlinear because of the quadratic advection term, which couples all scales of motion. Apparently, this is an underpinning reason for the cascade universality mentioned in the preceding paragraph. However, the "effective degree of nonlinearity" of the small-scale dynamics is not always quadratic and differs from one system to another. The implication is that the presumed cascades would have fundamental differences and would not be universal in a strict sense. For an example of the discrepancy in the effective degree of nonlinearity among fluid systems, let us consider the respective evolution equations for the 3D vorticity $\boldsymbol{\omega}$ and 2D vorticity gradient $\nabla\omega$ given by

$$\partial_t\boldsymbol{\omega}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u},\quad \nabla\cdot\boldsymbol{u}=0 \tag{1}$$

and

$$\partial_t\nabla\omega+(\boldsymbol{u}\cdot\nabla)\nabla\omega=\omega\boldsymbol{n}\times\nabla\omega-(\nabla\omega\cdot\nabla)\boldsymbol{u},\quad \nabla\cdot\boldsymbol{u}=0, \tag{2}$$

where $\boldsymbol{u}$ is the fluid velocity and $\boldsymbol{n}$ is the normal to the fluid domain in 2D. The stretching term $(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$ for the 3D vorticity $\boldsymbol{\omega}$ in Eq. (1) is essentially quadratic in $\boldsymbol{\omega}$ because the velocity gradient $\nabla\boldsymbol{u}$ is expected to behave as $\boldsymbol{\omega}$ on phenomenological grounds. As a consequence, an explosive 3D vorticity growth from a smooth initial vorticity field is possible, if not inevitable [5,6]. In contrast, the stretching term $(\nabla\omega\cdot\nabla)\boldsymbol{u}$ for the 2D vorticity gradient $\nabla\omega$ in Eq. (2) is virtually linear in $\nabla\omega$ because $\nabla\boldsymbol{u}$ is well behaved in the sense that the mean-square vorticity $\langle\omega^2\rangle=\langle|\nabla\boldsymbol{u}|^2\rangle$ is conserved. (Note that the rotation term $\omega\boldsymbol{n}\times\nabla\omega$ does not affect the amplitude of $\nabla\omega$.) As a result, the growth of 2D vorticity gradients can possibly be approximately exponential in time only, a relatively mild behavior. Hence, one would expect profound differences between the (highly nonlinear) 3D vorticity and the (nearly linear) 2D vorticity gradient dynamics. A notable example of these differences is that in the inviscid limit, the 2D enstrophy dissipation rate vanishes [7–9], whereas the 3D energy dissipation rate presumably remains nonzero. Another example is the discrepancy in the dependence on the Reynolds number of the number of degrees of freedom in the two cases [10].

The effective degree of nonlinearity in the above sense differs not only between 2D and 3D fluids but also among 2D fluid systems. In this study, we investigate this varying degree among members of a broad family of generalized models of 2D turbulence, first introduced by Pierrehumbert *et al.* [11]. By doing so, we extend several previous studies [12–15], aiming to unify our understanding of turbulent transfer in physically realizable fluid systems. The family's dynamics are characterized by the material conservation of the active scalar $\theta=(-\Delta)^{\alpha/2}\psi$, whose variance $\langle\theta^2\rangle$ is preferentially transferred to high wave numbers (small scales). Here ψ is the stream function, Δ is Laplace's operator, and α is a positive number. As the transfer of $\langle\theta^2\rangle$ proceeds to ever-smaller scales, the gradient $\nabla\theta$ grows without bound. This growth is due to the stretching term $(\nabla\theta\cdot\nabla)\boldsymbol{u}$, whose effective degree of nonlinearity depends on α and is wide ranging, from approximately linear to highly superlinear. Linear behavior is realized when $\nabla\boldsymbol{u}$ is a quantity of no smaller scales than θ, so that the transfer of $\langle\theta^2\rangle$ to the small scales (direct transfer) has no significant effects on $\nabla\boldsymbol{u}$. In other words, θ behaves nearly passively. This case corresponds to $\alpha\geq 2$, for which $\nabla\theta$ can grow approximately exponentially in time without acceleration. For $\alpha<2$, superlinear dynamics can be realized as the direct transfer of $\langle\theta^2\rangle$ entails a growth in $\nabla\boldsymbol{u}$, thereby, enhancing the production of $\nabla\theta$. This superlinearity reaches the familiar quadratic nonlinearity of three-dimensional turbulence at $\alpha=1$ and exceeds that for $\alpha<1$. The usual vorticity equation ($\alpha=2$) is the border line, where $\nabla\boldsymbol{u}$ and θ are of the same scale ($\langle|\nabla\boldsymbol{u}|^2\rangle=\langle\theta^2\rangle$), separating the linear and nonlinear regimes of the small-scale dynamics. We discuss these dynamical regimes in detail, with an emphasis on the local nature of the transfer of $\langle\theta^2\rangle$. The implication of the present results is that a comprehensive theory for this family of generalized 2D turbulence needs to account for the wide range of effective degrees of nonlinearity of the family's small-scale dynamics.

II. GOVERNING EQUATIONS

The equation governing the evolution of the family of active scalars $\theta=(-\Delta)^{\alpha/2}\psi$ (for $\alpha>0$) advected by the incompressible flow $\boldsymbol{u}=(-\psi_y,\psi_x)$ is

$$\theta_t+\boldsymbol{u}\cdot\nabla\theta=0. \tag{3}$$

This equation was proposed by Pierrehumbert *et al.* [11] in an attempt to better understand the nature of transfer locality in 2D turbulence, by examining how turbulent transfer responses to changes in the parameter α. Equation (3) is physically relevant for selected values of α. The usual 2D vorticity equation corresponds to $\alpha=2$. When $\alpha=1$, Eq. (3) is known as the surface quasigeostrophic equation and governs the advection of the potential temperature, which is proportional to $\theta=(-\Delta)^{1/2}\psi$, on the surface of a quasigeostrophic fluid. In addition to the genuine interest due to this physical significance [12–20], the surface quasigeostrophic equation has received some special attention for its resemblance to the 3D Euler system [21–24]. A mathematical feature of particular interest is the possible development of finite-time singularities (from smooth initial conditions), which, as argued by pioneering studies [21,22,25] of this problem, could be associated with the formation of weather fronts in the atmosphere. This, however, appears not to be the case [26].

For simplicity, we consider Eq. (3) in a doubly periodic domain of size L, and all fields concerned are assumed to have zero spatial average. This allows us to express the stream function as

$$\psi(\boldsymbol{x},t)=\sum_{\boldsymbol{k}}\hat{\psi}(\boldsymbol{k},t)\exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{4}$$

Here $\boldsymbol{k}=2\pi L^{-1}(k_x,k_y)$, where k_x and k_y are integers not simultaneously zero. The reality of ψ requires $\hat{\psi}(\boldsymbol{k},t)=\hat{\psi}^*(-\boldsymbol{k},t)$, where the asterisk denotes the complex conjugate. The fractional derivative $(-\Delta)^{\alpha/2}$ (which can be readily extended to $\alpha<0$, though not considered in this study) is defined by

$$\theta(\boldsymbol{x},t)=(-\Delta)^{\alpha/2}\psi(\boldsymbol{x},t)=\sum_{\boldsymbol{k}}k^{\alpha}\hat{\psi}(\boldsymbol{k},t)\exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}$$
$$=\sum_{\boldsymbol{k}}\hat{\theta}(\boldsymbol{k},t)\exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}, \tag{5}$$

where $k=|\boldsymbol{k}|$ is the wave number. Equation (3) expresses material conservation of θ, which gives rise to an infinite set of conserved quantities. In particular, the generalized enstrophy (active scalar variance)

$$Z=\frac{1}{2}\langle\theta^2\rangle=\frac{1}{2}\langle|(-\Delta)^{\alpha/2}\psi|^2\rangle=\frac{1}{2}\sum_{\boldsymbol{k}}k^{2\alpha}|\hat{\psi}(\boldsymbol{k},t)|^2 \tag{6}$$

is conserved. In addition, the generalized energy

$$E=\frac{1}{2}\langle\psi\theta\rangle=\frac{1}{2}\langle|(-\Delta)^{\alpha/4}\psi|^2\rangle=\frac{1}{2}\sum_{\boldsymbol{k}}k^{\alpha}|\hat{\psi}(\boldsymbol{k},t)|^2 \tag{7}$$

is also conserved. Note that E is the usual kinetic energy when $\alpha=2$, while Z is the usual kinetic energy when $\alpha=1$. Only for these cases is the kinetic energy conserved. The modal powers (spectra) of E and Z differ by the factor k^{α}. Therefore, the redistribution of a non-negligible amount of E to small scales would violate the conservation of Z. Similarly, the redistribution of a non-negligible amount of Z to large scales would violate the conservation of E. This means that if a spectrally localized profile is to spread out in wavenumber space, most of E and Z get transferred to large and small scales, respectively. This is the basis for the dual cascade hypothesis in 2D turbulence. Here we are mainly concerned with the direct transfer of Z. A more complete treatment should include the inverse transfer of E as well since these are known to be intimately related.

Given Eq. (4), we can express $\boldsymbol{u}=(-\psi_y,\psi_x)$ in terms of a Fourier series in the form

$$\boldsymbol{u}(\boldsymbol{x},t) = i\sum_{\boldsymbol{k}} (-k_y,k_x)\hat{\psi}(\boldsymbol{k},t)\exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{8}$$

By substituting Eqs. (5) and (8) into Eq. (3), we obtain the evolution equation for each individual Fourier mode $\hat{\theta}(\boldsymbol{k},t) = k^{\alpha}\hat{\psi}(\boldsymbol{k},t)$ of the conserved quantity θ

$$\frac{d}{dt}\hat{\theta}(\boldsymbol{k},t) = \sum_{\boldsymbol{\ell}+\boldsymbol{m}=\boldsymbol{k}} \frac{(m^{\alpha}-\ell^{\alpha})\boldsymbol{\ell}\times\boldsymbol{m}}{\ell^{\alpha}m^{\alpha}}\hat{\theta}(\boldsymbol{\ell},t)\hat{\theta}(\boldsymbol{m},t), \tag{9}$$

where $\boldsymbol{\ell}\times\boldsymbol{m}=\ell_x m_y-\ell_y m_x$. The sum on the right-hand side of Eq. (9) involves all modes [except $\hat{\theta}(\boldsymbol{k},t)$] and is a measure of the level of "excitation" of the mode $\hat{\theta}(\boldsymbol{k},t)$ due to all admissible wave vector triads $\boldsymbol{k}=\boldsymbol{\ell}+\boldsymbol{m}$. For a given triad, the coupling coefficient $(m^{\alpha}-\ell^{\alpha})\boldsymbol{\ell}\times\boldsymbol{m}/(\ell^{\alpha}m^{\alpha})$ depends on α. Its magnitude, together with the magnitudes of the coupling coefficients in the governing equations for $\hat{\theta}(\boldsymbol{\ell},t)$ and $\hat{\theta}(\boldsymbol{m},t)$, is a measure of triad dynamical activity, in the sense that larger (in magnitude) coupling coefficients correspond to more intense modal dynamics. This is intimately related to the effective degree of nonlinearity and locality of the small-scale dynamics as will be seen in the subsequent sections.

III. EFFECTIVE DEGREES OF NONLINEARITY OF THE SMALL-SCALE DYNAMICS

We now examine the behavior of $\nabla\theta$. Generally speaking, any derivative $(-\Delta)^{\eta}\theta$, for $\eta>0$, can be called a small-scale quantity. Here we consider $\nabla\theta$, which is a "twin brother" of $(-\Delta)^{1/2}\theta$, for its special status in Eq. (3) as well as its mathematical tractability. For $\alpha=2$, a similar treatment of $\Delta\theta=-\Delta\omega$ can be carried out in the same manner.

A. Growth of the active scalar gradient

The governing equation for $\nabla\theta$ is

$$\partial_t\nabla\theta + (\boldsymbol{u}\cdot\nabla)\nabla\theta = \nabla\times\boldsymbol{u}\times\nabla\theta - (\nabla\theta\cdot\nabla)\boldsymbol{u}, \tag{10}$$

which can be obtained by replacing ω in Eq. (2) by θ. Like Eq. (2), the effect of the first term on the right-hand side of Eq. (10) is to rotate $\nabla\theta$ without changing its magnitude. The amplification of $\nabla\theta$ is due solely to the stretching term $(\nabla\theta\cdot\nabla)\boldsymbol{u}$ and is governed by

$$\partial_t|\nabla\theta| + (\boldsymbol{u}\cdot\nabla)|\nabla\theta| = -\frac{\nabla\theta}{|\nabla\theta|}\cdot(\nabla\theta\cdot\nabla)\boldsymbol{u} \le |\nabla\boldsymbol{u}||\nabla\theta|. \tag{11}$$

Equation (11) implies that following the fluid motion, $|\nabla\theta|$ can grow exponentially in time with an instantaneous rate bounded from above by $|\nabla\boldsymbol{u}|$. Hence, the behavior of $|\nabla\boldsymbol{u}|$ holds the key to understanding the dynamics of $\nabla\theta$. Evidently, following the trajectory of a fluid "particle" starting from $\boldsymbol{x}=\boldsymbol{x}_0$ at $t=0$, the growth of $|\nabla\theta|$ is formally constrained by

$$|\nabla\theta| \le |\nabla\theta_0|\exp\left\{\int_0^t|\nabla\boldsymbol{u}|d\tau\right\}, \tag{12}$$

where $\theta_0=\theta(\boldsymbol{x}_0,0)$ and the integral is along the trajectory in question. Hence, on average, the rate r defined by

$$r = \frac{1}{t}\int_0^t|\nabla\boldsymbol{u}|d\tau \tag{13}$$

provides an upper bound for the exponential growth rate of $|\nabla\theta|$. Note that for $\alpha=1$ ($\langle|\nabla\boldsymbol{u}|^2\rangle=\langle|\nabla\theta|^2\rangle$), a double exponential growth of $|\nabla\theta|$ is allowed but not necessarily implied by the preceding equations. Nevertheless, it is interesting to note that Ohkitani and Yamada [24] observed such a behavior in their simulations, thereby, suggesting a negative answer to the question of finite-time singularities in the surface quasigeostrophic equation. This is consistent with the proof of nonexistence of blowup by Córdoba [26].

B. Linear versus nonlinear growth of $\nabla\theta$

Now for a sense of the behavior of r, we consider $\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$, which bounds $\langle|\nabla\boldsymbol{u}|\rangle$ from above by the Cauchy-Schwarz inequality $\langle|\nabla\boldsymbol{u}|\rangle\le\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$. For $\alpha\in[2,4]$, $\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$ can be estimated in terms of the inviscid invariants using the following version of the Hölder inequality (see, for example, Sec. 5 of Ref. [14]):

$$\begin{aligned}\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2} &\le \langle|(-\Delta)^{\alpha/4}\psi|^2\rangle^{1-2/\alpha}\langle|(-\Delta)^{\alpha/2}\psi|^2\rangle^{2/\alpha-1/2}\\ &= E^{1-2/\alpha}Z^{2/\alpha-1/2}.\end{aligned} \tag{14}$$

So $\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$ is controlled by the inviscid invariants E and Z. For $\alpha\notin[2,4]$, inequality (14) reverses direction. Furthermore, if an initial distribution of θ is to forever spread out in wave-number space, $\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$ increases without bound for this case. This implies that there exist different regimes of α, for which $\nabla\boldsymbol{u}$ evolves quite differently, and the active scalar gradient dynamics can be characteristically distinct. We discuss all these regimes in what follows.

For $\alpha<2$, the divergence of $\langle|\nabla\boldsymbol{u}|^2\rangle^{1/2}$ entails an accelerated growth of $\nabla\theta$ from an exponential one. This is the superlinear regime discussed in the introductory section. This superlinearity reaches the usual quadratic nonlinearity of 3D turbulence at $\alpha=1$, where $\langle|\nabla\boldsymbol{u}|^2\rangle=\langle|\nabla\theta|^2\rangle$. Hence, the surface quasigeostrophic and 3D Euler equations are analogous in this aspect. However, the analogy appears to be superficial as the surface quasigeostrophic equation turns out to be far more "manageable" than its 3D counterpart: a consequence of the material conservation of θ. For example, a number of global regularity results have been proved for the surface quasigeostrophic equation, by making use of mild dissipation mechanisms represented by $(-\Delta)^{\eta}$ with $\eta\ge 1/2$ [27–30], which can be much weaker than the usual viscosity. Whereas for the 3D Navier-Stokes system, viscosity appears to be inadequate for the same purpose. For $\alpha<1$, this quadratic nonlinearity is surpassed as the ratio $\langle|\nabla\boldsymbol{u}|^2\rangle/\langle|\nabla\theta|^2\rangle$ diverges in the limit $\langle|\nabla\theta|^2\rangle\to\infty$ because

$$\langle|\nabla\theta|^2\rangle^{2-\alpha} \le \langle|\nabla\boldsymbol{u}|^2\rangle\langle\theta^2\rangle^{1-\alpha} \tag{15}$$

(cf. Ref. [14]). Active scalar gradient production can then become highly intense.

For $\alpha\in[2,4]$, $\nabla\boldsymbol{u}$ is well behaved in the sense that its mean square is bounded from above in terms of the inviscid invariants [see Eq. (14)]. In this case, $\nabla\boldsymbol{u}$ is virtually unaffected by the direct transfer of $\langle\theta^2\rangle$. At large t, a general fluid trajectory is likely to have traversed the domain many times. The time average in Eq. (13) may therefore be approximately replaced by the spatial average. Hence, we can write

$$r \approx \langle|\nabla \boldsymbol{u}|\rangle \leq \langle|\nabla \boldsymbol{u}|^2\rangle^{1/2} \leq E^{1-2/\alpha}Z^{2/\alpha-1/2}, \tag{16}$$

where we have used the Cauchy-Schwarz inequality and Eq. (14). This approximation of r means that $\nabla\theta$ can grow exponentially in time without acceleration. Thus, approximately linear small-scale dynamics can be expected. Note that θ behaves almost as a passive scalar in this regime. The analogy between this case and that of a passive scalar was suggested by Schorghofer [12] on phenomenological grounds.

When $\alpha>4$, inequality (14) reverses direction, and $\langle|\nabla \boldsymbol{u}|^2\rangle^{1/2}$ can no longer be controlled by the inviscid invariants. However, unlike the case $\alpha<2$, for which $\langle|\nabla \boldsymbol{u}|^2\rangle^{1/2}$ diverges toward small scales, when $\alpha>4$ velocity gradients can be produced at increasingly large scales only. This production depends on the inverse transfer of the generalized energy E [14]. Within the direct transfer range, i.e., the generalized enstrophy range, the portion of $\langle|\nabla \boldsymbol{u}|^2\rangle$, say Ω, cannot increase and instead remains bounded from above in terms of Z. More precisely, as the spectra of $\langle|\nabla \boldsymbol{u}|^2\rangle$ and Z differ by the factor $k^{2\alpha-4}$, we have $\Omega \leq 2k_*^{4-2\alpha}Z$ (Poincaré type inequality), where k_* is the lower wave-number end of the generalized enstrophy range. This suggests that no significant changes in the effective degree of nonlinearity of the small-scale dynamics occur when α exceeds 4. Thus, we can expect approximately linear small-scale behavior for all $\alpha \geq 2$.

In passing, it is worth mentioning that while the small-scale dynamics appear to be insensitive to α in the regime $\alpha>2$, the large-scale dynamics can vary dramatically. The reason is that for large α, $\boldsymbol{u}$ is prone to divergence toward large scales as the inverse transfer of E proceeds. This undoubtedly intensifies motions at large scales. One may adapt the present notion of degree of nonlinearity for a quantitative measure of the large-scale dynamics. Analogous to the traditional problem of regularity, which is concerned with the possible divergence of $\nabla\theta$, there is a potential problem that $\boldsymbol{u}$ becomes divergent for sufficiently large α if the fluid is unbounded. This interesting problem is left for a future study.

IV. LOCALITY OF THE SMALL-SCALE DYNAMICS

This section is concerned with the small-scale dynamics at the modal level. We establish a connection between the degree of nonlinearity and dynamical activity of typical local triads at small scales. Here the dynamical activity of a given triad is associated with the magnitude of the coupling coefficients within the triad and is independent of the amplitude of the three modal members. These local triads are shown to be highly active for $\alpha<2$ and moderately active for $\alpha=2$ but become virtually inactive for $\alpha>2$. This implies that higher effective degrees of nonlinearity correspond to more dynamically intense local triads. Thus, the effective degree of nonlinearity is also a measure of dynamical activity of local triads at small scales. The transition at $\alpha=2$ from high activity to virtually no activity of local triads is consistent with phenomenological arguments [11] that the generalized enstrophy cascade is spectrally local for $\alpha<2$ but becomes dominated by nonlocal interactions for $\alpha>2$. Below, we also examine the dynamics of nonlocal triads and elaborate on the nature of the locality transition, in order to provide a detailed picture of the direct transfer of $\langle\theta^2\rangle$ at the modal level.

Within each individual triad $\boldsymbol{k}=\boldsymbol{\ell}+\boldsymbol{m}$, the transfer of modal generalized enstrophy is governed by

$$\begin{aligned}
\frac{d}{dt}|\hat{\theta}(\boldsymbol{k})|^2 &= \frac{(m^\alpha-\ell^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}}{m^\alpha\ell^\alpha} \\
&\quad\times[\hat{\theta}(\boldsymbol{\ell})\hat{\theta}(\boldsymbol{m})\hat{\theta}^*(\boldsymbol{k})+\hat{\theta}^*(\boldsymbol{\ell})\hat{\theta}^*(\boldsymbol{m})\hat{\theta}(\boldsymbol{k})] \\
&= C_{\boldsymbol{k}}[\hat{\theta}(\boldsymbol{\ell})\hat{\theta}(\boldsymbol{m})\hat{\theta}^*(\boldsymbol{k})+\hat{\theta}^*(\boldsymbol{\ell})\hat{\theta}^*(\boldsymbol{m})\hat{\theta}(\boldsymbol{k})], \\
\frac{d}{dt}|\hat{\theta}(\boldsymbol{\ell})|^2 &= \frac{(k^\alpha-m^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}}{k^\alpha m^\alpha} \\
&\quad\times[\hat{\theta}(\boldsymbol{k})\hat{\theta}^*(\boldsymbol{m})\hat{\theta}^*(\boldsymbol{\ell})+\hat{\theta}^*(\boldsymbol{k})\hat{\theta}(\boldsymbol{m})\hat{\theta}(\boldsymbol{\ell})] \\
&= C_{\boldsymbol{\ell}}[\hat{\theta}(\boldsymbol{k})\hat{\theta}^*(\boldsymbol{m})\hat{\theta}^*(\boldsymbol{\ell})+\hat{\theta}^*(\boldsymbol{k})\hat{\theta}(\boldsymbol{m})\hat{\theta}(\boldsymbol{\ell})], \\
\frac{d}{dt}|\hat{\theta}(\boldsymbol{m})|^2 &= \frac{(\ell^\alpha-k^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}}{\ell^\alpha k^\alpha} \\
&\quad\times[\hat{\theta}(\boldsymbol{k})\hat{\theta}^*(\boldsymbol{\ell})\hat{\theta}^*(\boldsymbol{m})+\hat{\theta}^*(\boldsymbol{k})\hat{\theta}(\boldsymbol{\ell})\hat{\theta}(\boldsymbol{m})] \\
&= C_{\boldsymbol{m}}[\hat{\theta}(\boldsymbol{k})\hat{\theta}^*(\boldsymbol{\ell})\hat{\theta}^*(\boldsymbol{m})+\hat{\theta}^*(\boldsymbol{k})\hat{\theta}(\boldsymbol{\ell})\hat{\theta}(\boldsymbol{m})],
\end{aligned} \tag{17}$$

where we have used the identities $\boldsymbol{\ell}\times\boldsymbol{m}=\boldsymbol{\ell}\times\boldsymbol{k}=\boldsymbol{k}\times\boldsymbol{m}$ and suppressed the time variable. It is well known that both E and Z are conserved for each individual triad. This can be readily verified by the fact that the coupling coefficients in Eqs. (17), $C_{\boldsymbol{k}}$, $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$, satisfy

$$C_{\boldsymbol{k}}+C_{\boldsymbol{\ell}}+C_{\boldsymbol{m}}=0=\frac{C_{\boldsymbol{k}}}{k^\alpha}+\frac{C_{\boldsymbol{\ell}}}{\ell^\alpha}+\frac{C_{\boldsymbol{m}}}{m^\alpha}.$$

Furthermore, the transfer of E and Z is from the intermediate wave number to both the larger and smaller wave numbers or vice versa (note the signs of the coupling coefficients). The former behavior appears to have been observed in numerical simulations of 2D turbulence without exception.

We now analyze the coupling coefficients $C_{\boldsymbol{k}}$, $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ in detail. As crude estimates that hold in general, these can be bounded by (assuming $k<l<m$)

$$\begin{aligned}
|C_{\boldsymbol{k}}| &= \frac{|(m^\alpha-\ell^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{m^\alpha\ell^\alpha} < k\ell^{1-\alpha}, \\
|C_{\boldsymbol{\ell}}| &= \frac{|(k^\alpha-m^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{k^\alpha m^\alpha} < \ell k^{1-\alpha}, \\
|C_{\boldsymbol{m}}| &= \frac{|(\ell^\alpha-k^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{\ell^\alpha k^\alpha} < \ell k^{1-\alpha}.
\end{aligned} \tag{18}$$

where we have used $|\boldsymbol{\ell}\times\boldsymbol{m}|=|\boldsymbol{\ell}\times\boldsymbol{k}|\leq k\ell$. Similar estimates were obtained in [20] (for $\alpha=1,2$) and in [31] (for $\alpha=1$). For $\alpha>2$, local triads (i.e., $k\lesssim\ell\lesssim m$) at small scales are effectively "turned off" because all $C_{\boldsymbol{k}}$, $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ tend to zero in the limit $k\to\infty$. Furthermore, the convergence is as rapid as $k^{2-\alpha}$. An immediate interpretation of this observation is that local triads can be relatively ineffective in the direct transfer of $\langle\theta^2\rangle$ compared with their nonlocal counterparts (see below). At the critical value $\alpha=2$, $C_{\boldsymbol{k}}$, $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ can remain order unity for local triads that satisfy $|\boldsymbol{\ell}\times\boldsymbol{m}|\approx k^2$ and $|m^\alpha-\ell^\alpha|\approx|k^\alpha-m^\alpha|\approx|\ell^\alpha-k^\alpha|\approx k^\alpha$. A majority of local triads satisfy both of these conditions. They are neither "ultrathin" nor nearly isosceles and correspond to relatively sharp estimates in Eqs. (18), which reduce to $|C_{\boldsymbol{k}}|\approx|C_{\boldsymbol{\ell}}|\approx|C_{\boldsymbol{m}}|\approx 1$. This means that local triads at small scales in the usual vorticity equation are moderately active. They can play a significant role in the direct transfer. Finally, for $\alpha<2$, the interaction coefficients of these same triads diverge as $k\to\infty$. Their divergence can be seen to be as rapid as $k^{2-\alpha}$. This result suggests that for this case, local triads can play an overwhelmingly dominant role in the direct transfer.

Next, we turn to nonlocal triads. These are thin triads with the wave numbers k, ℓ and m satisfying $k\ll\ell\lesssim m$. For this case, $C_{\boldsymbol{k}}$, $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ can be estimated as follows:

$$|C_{\boldsymbol{k}}|=\frac{|(m^\alpha-\ell^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{m^\alpha\ell^\alpha}\approx\frac{\alpha k^2}{\ell^\alpha},$$

$$|C_{\boldsymbol{\ell}}|=\frac{|(k^\alpha-m^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{k^\alpha m^\alpha}\approx\ell k^{1-\alpha},$$

$$|C_{\boldsymbol{m}}|=\frac{|(\ell^\alpha-k^\alpha)\boldsymbol{\ell}\times\boldsymbol{m}|}{\ell^\alpha k^\alpha}\approx\ell k^{1-\alpha}. \qquad (19)$$

In the limit $\ell\to\infty$ (while $k<\infty$), $C_{\boldsymbol{k}}$ vanishes, but both $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ ($C_{\boldsymbol{\ell}}\approx -C_{\boldsymbol{m}}$) diverge as rapidly as ℓ. This implies a vigorous exchange of generalized enstrophy between the two neighboring wave numbers ℓ and m, mediated by a virtually nonparticipating distant wave number k. This ultralocal transfer by nonlocal interactions is virtually independent of α as the divergence of $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ is insensitive to α. This result implies that local transfer by nonlocal interactions is an intrinsic characteristic of this family of 2D turbulence models. Note, however, that this transfer can be significant only when the spectrum of the generalized enstrophy is not steeper than k^{-1} [32]. In other words, the generalized enstrophy needs to be physically present at small scales in order to facilitate such a transfer. This suggests that for $\alpha>2$ (recall that local triads are dynamically inactive), the generalized enstrophy spectra can plausibly scale as k^{-1} because steeper spectra are unable to support a non-negligible direct transfer. This universal scaling was suggested by Schorghofer [12] and Watanabe and Iwayama [15]. Their justification is that θ can be considered as a passive scalar, a view in accord with the present analysis.

In passing, it is worth mentioning that the divergence of $C_{\boldsymbol{\ell}}$ and $C_{\boldsymbol{m}}$ in nonlocal triads is probably the reason for numerical instability in simulations of 2D turbulence with inadequate diffusion because local triads with coupling coefficients of order unity are evidently well behaved. Support for this claim can be derived from common observations that numerical divergences occur as soon as the modes in the vicinity of the truncation wave number are excited and well before they acquire any considerable amount of enstrophy. The same instability problem persists for $\alpha>2$, although the weak activities of local triads in this case may reduce the severity of the instability to a certain extent.

V. CONCLUDING REMARKS

We have presented the notion of effective degree of nonlinearity to quantify the small-scale dynamics of a family of generalized models of two-dimensional turbulence governed by a broad class of nonlinear transport equations. Here, the active scalar $\theta=(-\Delta)^{\alpha/2}\psi$ ($\alpha>0$) is advected by the incompressible flow $\boldsymbol{u}=(-\psi_y,\psi_x)$, where ψ is the stream function. We have argued that although the advection term is quadratic, the effective degree of nonlinearity of the small-scale dynamics is not always quadratic and depends on α. It has been found that the active scalar gradient dynamics are virtually linear for $\alpha\geq 2$ and become nonlinear for $\alpha<2$. Furthermore, the degree of nonlinearity increases as α is decreased from 2, becoming quadratic at $\alpha=1$ and exceeding quadratic nonlinearity for $\alpha<1$. It is conceivable that credible theories of the family's dynamics, particularly, those involving small scales, need to account for the dependence on α of the effective degree of nonlinearity.

We have also found that local triads at small scales are highly active for $\alpha<2$, moderately active for $\alpha=2$, and virtually inactive for $\alpha>2$. On the other hand, nonlocal triads are characterized by a vigorous exchange of generalized enstrophy between pairs of neighboring wave numbers, mediated by the third nonparticipating distant wave number. This property is common for all α, thereby, implying that nonlocal interactions (but ultralocal transfer) can be considered universal. In the absence of local triad activity ($\alpha>2$), this ultralocal transfer is responsible for the direct transfer of generalized enstrophy. This is similar to the problem of passive scalar transport by a large-scale flow as the weak feedback on the advecting flow by the active scalar can be neglected [32]. In this case, it appears plausible that generalized enstrophy spectra scale as k^{-1}.

The local nature of the generalized enstrophy transfer can be seen to be unambiguous in the present study. In general, this transfer is local in wave-number space regardless of what types of triads make the most contribution. For local triads, the generalized enstrophy transfer is inherently local. For nonlocal interactions, the transfer is even "more" local, having a relatively higher degree of locality compared to the transfer by local triads. More importantly, the transfer between distant wave numbers is largely insignificant. Hence, it makes sense to speak of the degree of locality of the direct generalized enstrophy transfer rather than to distinguish between local and distant transfers.

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